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A study of proportional reasoning: Tackling missing value and numerical comparison challenges

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Abstract.
This study aims to examine students' proportional reasoning in solving multiplicative problems. Three seven-grade students from SMPIT Al-Fahmi Palu were purposively selected using judgment sampling. Data analysis was conducted in three stages: data reduction, data presentation, and conclusion drawing/verification. The analysis was based on Bexter and Junker's theory of proportional reasoning, which consists of five stages: (1) qualitative, (2) early attempts at quantifying, (3) recognition of multiplicative relationships, (4) accommodating covariance and invariance, and (5) functional and scalar relationships. The results show that students with low ability (R_1) solved problems by recording the first measurement and pairing it with the second measurement through addition, indicating the early attempts at quantifying stage. Students with moderate ability (R_2) solved the problems by listing all possible combinations and summing them, indicating recognition of multiplicative relationships. Meanwhile, high-ability students (R_3) solved the problems by multiplying the first and second measurements, indicating they were in the accommodating covariance and invariance stage of proportional reasoning.

Keywords:
Proportional; reasoning; multiplicative problems

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INTRODUCTION

Proportional reasoning is a critical skill in mathematics that enables students to understand and solve problems involving relationships between quantities (Sari, 2024). The ability to recognize proportional relationships is fundamental not only in solving everyday practical problems but also in higher-level mathematical concepts. In many cases, students encounter two primary types of proportional reasoning tasks: solving for missing values in direct proportion and comparing numerical values across different contexts. These problems are common in both educational settings and real-life situations, such as scaling recipes, calculating distances, or understanding rates. Therefore, enhancing proportional reasoning skills is an essential goal in mathematics education.

Proportional reasoning is a fundamental cognitive skill that plays a crucial role in students' mathematical development, influencing their ability to solve problems involving ratios, rates, and proportions. Recent research highlights the importance of early exposure to proportional reasoning, suggesting that it not only develops in the formal stages of cognitive development but

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also through interventions that promote cognitive flexibility and categorization strategies. For example, a study by Scheibling-Sève et al. (2022) demonstrated that primary school students who received targeted training in multiple categorizations showed improved proportional reasoning skills and more diverse problem-solving strategies compared to a control group. This approach also helped bridge performance gaps related to socioeconomic status (SES), highlighting the potential of flexible thinking strategies in overcoming cognitive barriers to learning proportional reasoning.

In addition, a comprehensive analysis of proportional reasoning across various educational stages has identified specific challenges students face when applying this skill. Students often struggle with differentiating between proportional and non-proportional relationships, especially when confronted with real-life scenarios or abstract mathematical problems. Research by Ayan & Işıksal-Bostan (2019) emphasized that students frequently apply multiplicative methods to situations that require additive reasoning, which can hinder their understanding of when and how to apply proportional relationships correctly. These findings suggest that fostering a deeper understanding of the multiplicative nature of proportions through diverse problem types—such as missing value problems and qualitative reasoning—can significantly enhance students' proportional reasoning abilities

One of the most common problems related to proportional reasoning is the determination of missing values in a direct proportion. In these problems, students are asked to find an unknown quantity based on the relationship between two known quantities. For example, if 4 apples cost IDR 80.000, how much would 6 apples cost? This type of problem is encountered frequently in everyday scenarios, and it forms the foundation of many mathematical applications. The ability to recognize and apply the proportional relationship to solve for missing values is a vital skill that helps students gain confidence in their mathematical abilities.

Similarly, numerical comparison problems require students to understand how different quantities relate to each other in various measurement contexts. These problems can involve comparing prices, speeds, or distances across different units of measurement. For example, a student might be asked to compare the speed of two cars traveling at different speeds over different times. This type of reasoning is also critical in understanding rates and ratios, which are essential in fields such as economics, science, and engineering. Successful mastery of numerical comparison problems prepares students to make informed decisions based on quantitative data.

Although solving for missing values and numerical comparison seems straightforward, research suggests that students often struggle with these types of problems. Various factors contribute to these difficulties, including misconceptions about the underlying mathematical principles, a lack of familiarity with the types of problems encountered, and difficulties in translating real-world situations into mathematical representations. Furthermore, cognitive load theory suggests that when students face complex problems involving multiple steps, they may experience difficulty processing all of the relevant information and performing the necessary computations.

Recent research has highlighted that students often struggle with the concept of proportionality, especially when they face problems that deviate from the standard format of direct proportions (Mose & Case, 1999). For example, students may have difficulty applying proportional reasoning in more complex or non-standard problem formats, such as comparing quantities across different measurement spaces or multi-step problems. This issue becomes more pronounced when students are required to use advanced mathematical tools or interpret unfamiliar proportional relationships, which may lead them to apply inappropriate methods, resulting in incomplete or incorrect solutions.

These difficulties underscore the need for a deeper understanding of proportional reasoning that transcends basic, direct proportions. While students may grasp simple proportional relationships, they often fail to apply this understanding flexibly to more abstract or multi-step problems. As a result, they may struggle to connect different elements of the problem, causing confusion and errors. This challenge is especially evident in tasks such as product of measurement problems (PMP), where students must manipulate measurements across different units, requiring a more sophisticated application of proportional reasoning (Akatugba & Wallace, 1999).

Studies have emphasized the importance of targeted instructional strategies that promote flexibility in applying proportional reasoning. Teachers can guide students through diverse problem types, particularly those that involve multiple steps or more advanced mathematical tools, which fosters deeper conceptual understanding (Vasuki & Kumar, 2016). Such instruction enhances students' ability to approach complex proportional problems with confidence, ensuring they are better prepared for real-world applications

Ultimately, addressing the challenges identified in recent studies can bridge the gap between students' basic proportional reasoning skills and their ability to tackle more advanced mathematical problems, ensuring that students can effectively navigate complex scenarios across a variety of contexts. This type of comprehensive instruction is crucial for helping students build the problem-solving skills necessary for success in mathematics and beyond.

The challenges associated with proportional reasoning are not limited to the classroom. In real-world contexts, individuals often need to apply proportional reasoning when making decisions that involve financial, scientific, or social data. For instance, when calculating the best buy between two different brands of cereal, a person must use proportional reasoning to compare prices and quantities. Similarly, in scientific research, proportional reasoning is used to analyse data, determine relationships between variables, and draw conclusions from experimental results. These real-world applications highlight the importance of developing strong proportional reasoning skills early on.

Researchers have examined various methods to improve students' proportional reasoning abilities. One approach is the use of visual aids and manipulatives to help students understand the relationships between quantities. According to Lamon (2020), visual representations of proportions can help students develop a deeper understanding of the concept by allowing them to manipulate and explore different scenarios. Another method involves encouraging students to explain their reasoning verbally or in writing, which can help reinforce their understanding of the underlying concepts.

Despite these efforts, there is still much to be done to improve students' proportional reasoning skills, especially in tackling missing value and numerical comparison problems. It is essential for educators to understand the specific challenges students face when dealing with these types of problems and to develop targeted interventions to address these difficulties. By providing students with the tools and strategies they need to approach proportional reasoning problems effectively, educators can help them develop a more comprehensive understanding of mathematics and better prepare them for future challenges.

In light of these challenges, this study aims to explore how students tackle missing value and numerical comparison problems that involve proportional reasoning. By examining the strategies students use to solve these problems, the research will contribute to the understanding of the cognitive processes involved in proportional reasoning and identify potential areas for intervention. The findings of this study can inform instructional practices and help educators develop more effective teaching strategies that target the specific needs of students struggling with proportional reasoning.

In contrast, the product of measurement problem involves multiplying two measurements, with the result being a new measurement that differs from the original measurements. For example, in a problem involving hearts and spades cards, the multiplication of two measurements (the number of hearts and the number of spades) gives the number of possible card pairs, which is distinct from the original measurements (Devore, 2015). Initial observations showed that students could solve direct proportion problems using various methods, but they faced difficulties when solving product of measurement problems. Some even used incorrect reasoning, such as counting cards in a way that did not align with the problem's instructions.

The observations indicate that students' understanding of product of measurement problems is still lacking, as evidenced by the incorrect reasoning and methods that resulted in inaccurate solutions. One of the causes of this is the lack of experience students have in facing diverse multiplicative problems. Previous research by ÅžEN & GÃœLER (2018) examined the characteristics of proportional reasoning processes used by students in solving ratio and proportion problems Recent studies have examined how students approach proportional reasoning in more complex mathematical contexts, such as product of measurement problems (PMP), which require the manipulation of different units of measurement. These problems have been identified as particularly challenging for students because they require a deeper understanding of how multiplicative relationships function across measurement spaces. A study by Khasawneh et al. (2022) investigated how eighth-grade students solved problems involving proportional reasoning, emphasizing the difficulty they encountered when combining various units of measurement. Despite being able to solve simpler proportional problems, students often struggled with multi-step problems that involved multiple dimensions, such as those encountered in PMP tasks. These findings underscore the need for instructional strategies that enhance students' ability to apply proportional reasoning flexibly in more abstract and complex contexts.

Furthermore, research by Ojose (2015) explored students' misconceptions in proportional reasoning, particularly in tasks that deviate from the standard formats students are typically taught. Their study highlights how difficulties arise when students are faced with unfamiliar problem types that require proportional reasoning across diverse contexts. The study found that students often revert to incorrect methods when they lack a robust understanding of how proportionality works in real-world scenarios. These findings suggest that educators should provide opportunities for students to engage with diverse problem types, ensuring they develop a more comprehensive and adaptable understanding of proportional reasoning.

In recent years, there has been substantial research into the development of proportional reasoning, which continues to be foundational for mathematical problem-solving, particularly in areas like algebra, ratios, and fractions. The reasoning strategies employed by students in these studies are often grounded in frameworks that outline stages in the development of proportional reasoning (Hanna, 2020). One such framework involves five stages: (1) qualitative reasoning, where students focus on descriptive or qualitative judgments; (2) early attempts at quantifying, where students start using numbers but struggle with complex relationships; (3) recognition of multiplicative relationships, where students begin to apply multiplication as the primary operation in solving proportional problems; (4) accommodating covariance and invariance, which involves understanding how variables change while maintaining a constant relationship; and (5) functional and scalar relationships, where students can deal with abstract proportionality in advanced contexts

In a more recent study, Jeong & Huttenlocher (2007) explored how young children, as early as five years old, begin to show proportional reasoning abilities, revealing a variety of developmental stages and understanding that varied based on the nature of the quantities involved, such as discrete versus continuous quantities. Their findings suggest that proportional reasoning skills develop early in childhood, though mastery of symbolic proportional reasoning (such as with ratios and fractions) continues to improve throughout elementary school. The study emphasized the potential of interventions aimed at training proportional reasoning skills at a young age, especially through activities that involve continuous quantities and interactive play, which have shown positive effects on students' ability to engage with proportional problems later on. These insights are important for shaping how educators can support students in transitioning from basic proportional tasks to more complex, abstract mathematical reasoning (Wilson et al., 2013).

By adopting Baxter and Junker's stages of proportional reasoning, the current study aims to track and categorize the reasoning strategies employed by eighth-grade students as they solve multiplicative problems, particularly those involving product of measurement. This approach provides a structured way to assess how students' progress in their understanding of proportionality and how they apply these concepts to solve real-world mathematical problems. The stages provide a lens through which the researcher can observe how students transition from one level of reasoning to another, identifying key moments of growth or struggle in their cognitive development.

The use of this theoretical framework also facilitates a more nuanced understanding of students' abilities, allowing for a detailed analysis of the specific challenges they encounter as they solve complex problems involving proportional reasoning. By comparing students' performance at different stages, the study can offer insights into which strategies are most effective at each stage of reasoning and how instruction can be tailored to meet students at their current level of understanding. Ultimately, the application of Baxter and Junker's stages of proportional reasoning provides valuable insights into the cognitive processes involved in solving multiplicative problems, particularly in the context of product of measurement problems, and offers a foundation for improving instructional practices in mathematics education.

The novelty of this research lies in its focus on exploring the cognitive processes involved in solving proportional reasoning problems, particularly those involving missing values and numerical comparison, through a detailed examination of students' strategies. By integrating Baxter and Junker's stages of proportional reasoning, the study offers a new approach to tracking and categorizing students' progression in understanding complex proportional problems, such as product of measurement problems (PMP). This framework allows for a deeper insight into how students transition through different stages of reasoning, revealing key moments of growth or struggle. Additionally, the research emphasizes the importance of addressing misconceptions and challenges students face with non-standard proportional problems, offering potential intervention strategies to help students overcome difficulties and apply proportional reasoning more flexibly in both mathematical and real-world contexts.

METHOD

This study is qualitative research aiming to explore students' problem-solving processes involving proportional reasoning and multiplication principles, focusing on their ability to solve measurement product problems. It seeks to identify how students with different mathematical abilities approach similar mathematical problems and to uncover the reasoning and strategies they employ. A qualitative approach was chosen to gain a deeper understanding of students' thought processes and problem-solving strategies.

The study involved seventh-grade students from SMPIT Al-Fahmi Palu, with an initial sample of 26 students who were given problem-solving tasks designed to assess their ability to apply multiplication principles and proportional reasoning. These tasks aimed to evaluate how students' approach and solve mathematical problems requiring critical thinking and logical reasoning. From this initial group, three students were purposively selected to represent different levels of mathematical ability: high, medium, and low. This selection ensured a comprehensive analysis of various problem-solving approaches and strategies employed by students with varying levels of mathematical competence.

The research followed a systematic procedure to ensure the collection of high-quality and relevant data. The first stage involved administering problem-solving tasks to all 26 students to assess their ability to apply multiplication principles and proportional reasoning in real-world scenarios. The second stage involved selecting three students for in-depth analysis based on their mathematical proficiency. The third stage consisted of semi-structured interviews where the selected students were asked to explain their thought processes while solving the given problems. These interviews aimed to explore the strategies, challenges, and reasoning behind their problem-solving approaches. Additionally, non-verbal observations were conducted to analyze students' facial expressions and body language during problem-solving activities, providing further insights into their cognitive and emotional engagement with the tasks.

The primary research instrument consisted of problem-solving questions related to measurement products that tested students' application of multiplication principles and proportional reasoning. One example of the problems used in this study is: "If 3 meters of fabric costs 75,000 IDR, how much would 5 meters of fabric cost?" This question was designed to assess students' ability to apply proportional reasoning in a practical context. The problems were carefully designed to align with the curriculum and the mathematical abilities targeted in this research.

Furthermore, the questions were structured with varying levels of difficulty to ensure the inclusion of all student competency levels.

Data collection methods included interviews, non-verbal observations, and audio/video recordings. The interviews were conducted to obtain verbal explanations of students' problem-solving processes, allowing them to articulate the steps and strategies they used to arrive at their answers. Non-verbal observations were carried out to examine students' facial expressions and body language, which provided additional insights into their comprehension and cognitive engagement with the problem-solving tasks. All interviews were recorded to ensure data accuracy, and transcripts of these recordings were later analyzed to identify patterns in students' reasoning and mathematical problem-solving strategies.

The data analysis process followed the framework developed by Miles, Huberman, and Saldaña (2016), consisting of three main steps: data reduction, data presentation, and conclusion drawing/verification. The first step, data reduction, involved categorizing interview transcripts based on indicators of proportional reasoning found in students' explanations. Any irrelevant or excessive data were eliminated to focus on key findings. The second step, data presentation, involved organizing the remaining data into a clear and structured format, allowing the identification of patterns and relationships within the data. The final step, conclusion drawing and verification, involved interpreting the presented data to derive meaningful insights. The validity of these conclusions was ensured through multiple methods, such as revisiting field notes, conducting peer reviews, and comparing findings with existing literature. This verification process ensured that the conclusions drawn were consistent and aligned with the collected data.

By employing this methodology, the study aims to provide a deeper understanding of how students with different levels of mathematical ability solve problems involving proportional reasoning and multiplication. The combination of interviews, non-verbal observations, and qualitative data analysis allows for a comprehensive exploration of students' cognitive processes in mathematical problem-solving. The findings from this study are expected to offer valuable recommendations for developing more effective and adaptive teaching strategies that cater to the diverse needs of students.

RESULTS AND DISCUSSION

This study aims to describe the proportional reasoning of eighth-grade students in solving multiplicative problems of the product of measurement type, focusing on the problem-solving strategies employed by the students. A product of measurement problem (PMP) is a type of mathematical problem that involves multiplication between two distinct measurement spaces, resulting in a third measurement space. In these problems, the two measurement spaces are treated as independent entities, and the problem is solved by multiplying values from each space to determine the final outcome. According to the definition, a PMP does not involve direct proportion; rather, it involves two separate proportional relationships that are linked through multiplication, creating a more complex scenario than simple proportional reasoning.

The complexity of the PMP lies in the requirement for students to apply their understanding of multiplication across different units of measurement, which may not be directly related in a proportional sense. This type of problem tests students' ability to manage and manipulate multiple variables and measurement spaces simultaneously. The study investigates how students approach these problems, specifically how they use proportional reasoning to navigate the multiplication of different measurement spaces. By focusing on the strategies employed by students, the study seeks to identify the various ways in which students comprehend and solve these types of problems, highlighting the different stages of understanding and application of proportional reasoning.

To gain a deeper understanding of students' problem-solving processes, the study uses a specific product of measurement problem (PMP) as an example, which is provided to the students for analysis and solution. The chosen problem allows the researchers to observe how students interpret the relationships between the different measurement spaces and how they use multiplication to link these spaces together. This problem serves as a practical example of how students engage with mathematical concepts such as multiplicative relationships, measurement, and proportional reasoning. By analyzing their responses, the study can provide insights into the cognitive strategies students use to solve these problems and identify any misconceptions or challenges they may face in applying these mathematical principles.

Ultimately, the goal of the study is to capture a detailed picture of how eighth-grade students' reason proportionally when faced with a product of measurement problem. By examining the problem-solving strategies employed by the students, the study aims to contribute to the broader understanding of how students develop and apply proportional reasoning, particularly in contexts that require the multiplication of different measurement spaces. The findings from this study could help inform instructional practices and strategies, offering valuable insights for educators seeking to improve students' understanding of proportional relationships and enhance their ability to solve complex mathematical problems involving measurement and multiplication.

Dina has 5 types of postcards and 4 types of stamps. The postcards each feature pictures of flowers, landscapes, animals, landmarks, and food. The stamps come in blue, red, green, and black. Dina will choose 3 postcards and 1 stamp to combine and send to her friends. How many different ways can Dina select 3 postcards and 1 stamp?

The problem-solving strategies used by the students were categorized into five stages based on the theory of building proportional reasoning by Bexter and Junker. These stages are: (1) qualitative, (2) early attempts at quantifying, (3) recognition of multiplicative relationships, (4) accommodating covariance and invariance, and (5) functional and scalar relationships. Table 1 is an explanation of each of these five stages in developing proportional reasoning.

Table 1. Stages of students' proportional reasoning

Stage of Proportional Reasoning	Description of proportional reasoning stages	Proportional Reasoning in Action
1. Qualitative	Students have a lot of knowledge about quantities that allow them to answer questions about fairness (dividing something fairly) or questions about more and less (comparing things). Example of a more and less question: Which drink is sweeter? Example of a fairness question: How can you divide a pizza so that each child gets an equal portion?	• Students can solve problems involving dividing something fairly • Students can solve problems involving comparing things. • Students use guessing as a method to find an answer.
2. Early attempts at quantifying	Early attempts in measurement often involve differences in constants through addition rather than using multiplicative relationships. Students still rely on calculations using addition or subtraction	• Students use a solution method that involves constant addition differences of a quantity rather than using multiplicative relationships • Students still use calculations that continually decrease or increase. • Students still use calculations that continually decrease or increase
3. Recognition of multiplicative relationships	Students have the intuition that a ratio consists of two numbers that change Together, but the change may result from either addition or multiplication. Students tend to use addition	• Students have the intuition that a ratio involves two numbers that change together; however, they still think that the change May result from either addition or multiplication. • Students tend to use additive strategies more frequently when faced with multiplication situations

Stage of Proportional Reasoning	Description of proportional reasoning stages	Proportional Reasoning in Action
4. Accommodating covariantce and invariance	Students begin to develop a model of multiplicative change. They realize that when several quantities may change, the relationship between those quantities remains invariant. Students view the ratio as a unit of measure that can be used. Students are able to distinguish between situations involving absolute change and situations involving relative changes. Multiplication strategies are used in certain contexts or problems, but if students are faced with a challenging context, they will return to using addition reasoning.	• Students begin to develop a solution model, which means solution model, which means that when several quantities change, the relationship between those quantities remains the same. • Students seek the unit factor and apply multiplication approaches. • •Students use scalars to build the model they develop. • Students can find the unit value of an existing measurement and use it to solve the problem as a whole. • When students fail to build a solution model, they return to the method of determining differences in addition.
5. Functional and scalar relationships	Students understand the invariant relationship between changing quantities. They have a general model for solving problems and choose efficient strategies to use. Students have a good understanding of the concepts of covariance and invariance.	• Students understand the consistent relationship between changing quantities. • Students have a general model for solving problems and choose efficient strategies to use. • Students understand the structure of relationships in each measurement.

Recent research in mathematics education highlights that students employ various reasoning strategies when solving proportional problems. Studies indicate that some students struggle to distinguish between multiplicative and additive relationships in proportional reasoning, leading to errors in their solutions (Tan Sisman & Aksu, 2016). Additionally, some students rely on nonproportional strategies, such as incorrect rule-based approaches, which often result in conceptual misunderstandings (Khalid et al., 2018). These errors may include an inability to identify the appropriate mathematical operations and a tendency to apply incorrect procedures without a deep conceptual understanding. Consequently, instructional approaches should emphasize conceptual learning by integrating contextual experiences and manipulatives, allowing students to build stronger proportional reasoning skills (Sisman & Aksu, 2016; Khalid et al., 2018).

In contrast, R_2 demonstrates a partial understanding of the product of measurement (MPM) concept by recognizing that it involves multiplicative relationships. However, despite this awareness, R_2 struggles to correctly identify and apply these relationships in each component of the problem. Based on recent research on proportional reasoning development, students at this stage begin to recognize multiplicative structures but lack the ability to apply them consistently (Bartell et al., 2015). This observation aligns with findings by recent studies, which suggest that students often attempt to use multiplication but revert to simpler additive strategies when they fail to recognize proportional relationships. This pattern is evident in R_2's attempt to determine the number of possible combinations of three postcards and one stamp by listing all unit elements and summing them rather than applying the correct multiplicative approach.

Furthermore, research by Bartell et al. (2015) indicates that students at this level employ informal reasoning about proportional situations, often using models or manipulatives to represent mathematical relationships. While this approach demonstrates some conceptual understanding, it also highlights gaps in mastery, as students rely on trial-and-error methods rather than systematically applying proportional reasoning. The use of informal strategies suggests that while

R_2 is beginning to grasp proportionality, additional instructional support is required to help them transition from intuitive reasoning to formalized mathematical thinking. These findings emphasize the need for structured learning interventions to reinforce multiplicative reasoning, thereby enabling students to move beyond reliance on additive strategies and develop more sophisticated problem-solving techniques.

R_3 solves the MPM problem by combining addition and multiplication strategies. R_3 uses addition to determine the number of two-postcard combinations that can be made from four different postcards (P_1). Then, R3 identifies the multiplicative relationship between (P_1) and (P_2) to determine (P_3). Based on the stages of developing proportional reasoning proposed by recent research on proportional reasoning development, this process aligns with the stage of accommodating covariance and invariance, where students begin to understand the proportional relationships between quantities. Further studies indicate that students at this level begin forming a conceptual model of change and can predict when a quantity will vary proportionally. This is evident when R_3 justifies the reasoning behind the model $P_3 = P_1 \times P_2$. Research also classifies R_3's strategy under informal reasoning about proportional situations and quantitative reasoning. Informal reasoning is demonstrated when R_3 finds P_1 using a listing strategy, a common approach among students with developing proportional reasoning abilities (. The quantitative reasoning level appears when R_3 calculates P_3, thinking that if the number of stamps is 1, then $P_3 = 10 \times 1 = 10$; if there are 2 stamps, then $P_3 = 10 \times 2 = 20$; and if there are 4 stamps, then $P_3 = 10 \times 4 = 40$. R_3 demonstrates reasoning at multiple levels compared to previous research because the problems used in other studies typically involve direct proportion problems rather than complex proportional structures. However, in this study, the problems used are product of measurement problems, which involve Cartesian multiplication between two measurement spaces within a third measurement space (Tillema, 2013).

The problem requires Dina to choose 3 postcards from 5 different types and 1 stamp from 4 different types, then combine them to create unique sets to send to her friends. To determine the total number of possible ways Dina can complete this task, it is necessary to apply the mathematical concepts of combinations and multiplication of possible outcomes. Since the order in which the postcards are selected does not affect the final outcome, the combination formula is used to calculate the number of ways to choose 3 postcards from 5. This ensures that all possible groupings of postcards are considered without redundancy. Once the number of postcard combinations is determined, the next step is to consider the selection of the stamp. Since each unique combination of postcards can be paired with any of the 4 different stamp options, the fundamental principle of multiplication is applied. This principle allows us to find the total number of possible outcomes by multiplying the number of postcard combinations by the number of stamp choices, ensuring that all potential pairings are accounted for systematically.

Understanding the application of combinations and multiplication in this context highlights the importance of proportional reasoning in problem-solving. The use of combinations ensures that selections are made without repetition, while multiplication accounts for independent choices that expand the number of possible outcomes. This approach reinforces key mathematical principles that are essential for solving similar combinatorial problems, where multiple elements must be selected and paired systematically. By breaking down the problem into distinct stages—first determining the number of postcard groupings and then considering the stamp selection—students develop a structured approach to problem-solving that strengthens their analytical thinking. Furthermore, mastering these mathematical strategies helps in understanding more advanced concepts in probability and discrete mathematics, demonstrating the practical applications of proportional reasoning beyond simple counting.

The first step is to calculate the number of ways Dina can choose 3 postcards out of 5 types of postcards. where n is the total number of items, and r is the number of items to be selected. Substituting $n = 5$ and $r = 3$. Thus, there are 10 different ways to select 3 postcards from 5. Next, we calculate the number of ways Dina can choose 1 stamp out of 4 types of stamps. Since only one stamp is being selected, and there are no additional conditions, there are simply 4 ways to choose a stamp.

Now, we apply the multiplication rule of counting, which states that if one event can occur in m ways and another independent event can occur in n ways, then the total number of ways both events can occur together is given by multiplying m and n (Lockwood & Purdy, 2019). Therefore, the total number of ways Dina can choose and combine 3 postcards with 1 stamp is total combinations $= 10 \times 4 = 40$. This means Dina has 40 different ways to select and combine 3 postcards and 1 stamp. Combinations are used when the order of selection does not matter, which is the case when Dina selects the postcards. Additionally, we apply the multiplication rule of counting, which is useful for determining the total number of possible outcomes when multiple independent events are involved.

This problem can also be associated with challenges related to Missing Value and Numerical Comparison. The Missing Value Challenge occurs when the total number of possibilities is not explicitly provided, which requires students to infer or calculate the value based on the available data and relevant mathematical principles. In this case, the missing value refers to the total number of possible combinations, which must be determined by first calculating the number of ways to combine postcards and stamps separately. Once these values are identified, the next step involves multiplying them to find the total number of combinations. This process forces students to engage in logical reasoning, applying mathematical operations systematically to derive the missing value, thus enhancing their problem-solving abilities.

Moreover, addressing the Missing Value Challenge fosters a deeper understanding of proportional reasoning and combinatorial thinking. As students work through the steps of calculating combinations and multiplying values, they strengthen their ability to analyze relationships between different elements. This is a crucial aspect of mathematical reasoning, as it involves understanding how different quantities can interact and combine in various ways. Beyond solving this particular problem, this challenge prepares students for more complex mathematical tasks in which they must identify missing information and make logical connections. Ultimately, the process of addressing missing values not only improves students' problem-solving skills but also deepens their comprehension of broader mathematical concepts, which are fundamental in many areas of study.

The Numerical Comparison Challenge is relevant because solving this problem requires comparing and operating on different numerical quantities. Specifically, it involves analyzing the possible combinations of postcards and the available choices of stamps, then multiplying these values to determine the total number of possible outcomes. This process highlights the importance of understanding numerical relationships and making accurate comparisons to reach a correct solution. Mastering numerical comparison is essential for solving similar mathematical problems, as it helps develop critical thinking skills and enhances students' ability to work with proportional reasoning, multiplication principles, and combinatorial analysis. By effectively comparing and combining numerical values, students gain a deeper understanding of mathematical structures and improve their overall problem-solving abilities.

CONCLUSIONS

Based on the results and discussion presented earlier, it can be concluded that the mathematical thinking process of eighth-grade students in solving product of measurement problems is categorized into five stages of developing proportional reasoning, as proposed by Bexter and Junker: (1) qualitative, (2) early attempts at quantifying, (3) recognition of multiplicative relationships, (4) accommodating covariance and invariance, and (5) functional and scalar relationships. The following conclusions can be drawn from this study: According to Bexter and Junker's theory, the proportional reasoning of Student 1 (R_1) falls under the stage of early attempts at quantifying, where initial efforts in measurement often involve constant additive differences rather than using multiplicative relationships. At this stage, students still rely on calculations based on addition or subtraction. The student with intermediate ability (R_2) solved the problem by listing all possible combinations and then adding them up. R_2 was aware that the product of measurement problem could be solved using a functional approach but failed to recall how to apply the method correctly. Therefore, R_2 opted to list all possible combinations and sum them accordingly. Based on the description, it was found that the student used an additive approach by summing the individual units of possible combinations of 3 postcards and 1 stamp. The student recognized that the problem could potentially be solved using a functional formula but was unable to determine the relationship between the elements in each measurement space, leading them to choose an addition strategy instead. According to Bexter and Junker's theory, R_2's proportional reasoning falls under the "recognition of multiplicative relationships" stage. At this stage, students have an intuition that a ratio involves two quantities that change together, but the change may be interpreted as resulting from either addition or multiplication. As a result, students tend to use additive strategies even when multiplicative reasoning is expected. Students with high ability (R_3) solved the problem by listing the number of first measurements (P_1) and then multiplying it by the second measurement (P_2). Based on the explanation, it is known that the student used multiplication to determine the total number of possible combinations. The student employed a method of identifying unit elements and summing these unit elements to find P_2, commonly referred to as missing a value. According to Bexter and Junker's theory, R_3's proportional reasoning is at the stage of accommodating covariance and invariance. At this stage, the student begins to develop a model of multiplicative change. The student understands that when certain quantities may change, the relationship between these quantities remains invariant (constant). The student applies a multiplication strategy in specific contexts or problems; however, when faced with more complex contexts, the student reverts to additive reasoning, as seen in R_3's approach when determining P_1.

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